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The Decision Problem for Effective Procedures

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An *(effective) decision procedure for* property F *applicable to* things of kind *K* (a "decision procedure for Fness of *K*s") is an effective procedure (effective method) for ascertaining, concerning any input of *K*, whether it has F or instead lacks it. A decision procedure for a property F applicable to a kind *K* (*effectively*) *decides* a class G iff G is the class of things of kind *K* having the decision procedure's target property F. A class G is (*effectively*) *decidable* ("the Gs are decidable") iff there exists a (known or unknown) decision procedure that decides G; otherwise G is *undecidable*. *The decision problem* for a property F and a kind *K* is the problem of producing a decision procedure for Fness of *K*s. A decision problem is *solvable* iff there exists a (known or unknown) decision procedure that solves it, *unsolvable* otherwise.

The decision problem for effective procedures: *Produce a decision procedure for whether a given procedure is, or is not, effective (algorithmic).*

An *ascertainment procedure for* the *wh*-facts of a certain class is a procedure for determining the answers to *wh*-questions of that class. A *valuation procedure* for a function is an ascertainment procedure for the "what value" facts of that function for a given argument. A *judgment procedure for* a class of propositions is a "whether" ascertainment procedure for that class. A *judgment procedure for* property F *applicable to* (things of) kind *K* ("a judgment procedure for Fness of *K*s") is a judgment procedure for the proposition that a given input of *K* has F. A judgment procedure for Fness of *K*s *certifies* all and only those things of *K* that it deems to have F. A judgment procedure is *self-certifying* iff it certifies itself, and is *non-self-certifying* otherwise.

An *effective calculation procedure* for a function is an effective valuation procedure for the function. A function is *effectively calculable* iff there exists a (known or unknown) effective calculation procedure for it. A decision procedure for Fness of *K*s is an effective judgment procedure for Fness of *K*s. A decision procedure P^{K}_{F} for Fness of *K*s immediately yields its mirror image—a decision procedure $P^{K}_{\sim F}$ for the complement property. A judgment procedure is *effectively self-certifying* if it is both a decision procedure and self-certifying; otherwise it is *non-effectively-self-certifying*. A judgment procedure is *effectively non-self-certifying* iff it is both a decision procedure and non-self-certifying; otherwise it is *non-effectively-non-self-certifying*.

The effective self-certification lemma: *The class of self-certifying decision procedures is undecidable.* ***Proof:*** Any decision procedure for effective self-certification of judgment procedures would immediately yield its mirror image, a decision procedure for non-effective-self-certification of judgment procedures. There is no such thing.

$P^{JP}{}_{SC}$: 1. Given an input judgment procedure P^{JP} that is applicable to judgment procedures, determine whether P^{JP} is a decision procedure (effective for some property or other); 2. If P^{JP} is not a decision procedure, conclude that P^{JP} is not effectively self-certifying; 3. If P^{JP} is a decision procedure, apply it to itself; 4. If the result of self-application is negative, or there is no unique, unequivocal yes/no result of self-application, conclude that P^{JP} is not effectively self-certifying. 5. Iff the result of self-application is positive, conclude that P^{JP} is effectively self-certifying.

The undecidability of effectiveness: *The class of effective procedures (algorithms) is undecidable.* ***Proof:*** None of steps 2-5 disqualify $P^{JP}{}_{SC}$ from being effective. If the effective ascertainment procedures were decidable, then $P^{JP}{}_{SC}$ as a whole would represent a specific decision procedure, one of the very sort precluded by the effective self-certification lemma.