Abstract

There is a fundamental duality that runs through the exact sciences. At the logical level, it is the duality between (Boolean) logic of subsets and the logic of partitions. The quantitative versions of the dual logics are logical probability theory and logical information theory. The duality accounts for the duality in the category of *Sets* and its opposite $Sets^{op}$. The partial order in the two dual logics gives the two fundamental canonical functions and the claim is that all canonical morphisms in *Sets* arise from those two morphisms. In physics, there is the notion of "definiteness all the way down" which arises in classical physics and dually there is the notion of definiteness only down to a certain level and then objective indefiniteness that arises in quantum physics.

A Fundamental Duality in the Exact Sciences:

An Introduction to Mathematical Metaphysics

David Ellerman
University of Ljubljana

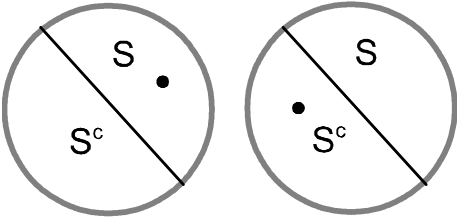
A Fundamental Duality –Overview: I

Fund. Duality	Subset side "Its"	Partition side "Dits"
Logic	Logic of subsets	Logic of partitions
Quant. Logic	Logical Probability	Logical Information
CT Duality	Subobjects & limits	Quot. Obj. & colimits
CT Canonicity	If $S \subseteq T$, $S \rightarrowtail T$	If $\sigma \precsim \pi$, $\pi \twoheadrightarrow \sigma$
Creation	Ex Nihilo creation	Big Bang creation
Alg. vs. Geom.	quiddity with haecceity	quiddity. w/o haecceity
Physics	Fully def. Cl. Mech.	Indef. Q. Mechanics
Biology	Selectionist Mech.	Generative Mechanism

The "Big Picture" of Mathematical Metaphysics

–

Its: On the elements- or Its-side of the duality, the relevant question about a *single* element $\{\bullet\}$ is existence versus nonexistence, e.g., an element is either in a subset or in the complementary subset, i.e., to be or not to be.
Logic of elements is Boolean logic, an element is either in S or S^c.



A Fundamental Duality –Overview: III

Dits: On the distinctions- or Dits-side of the duality, the relevant question about a *pair* of elements $\{\bullet, \star\}$ is distinction or indistinction, e.g., an ordered pair of elements is either a distinction or indistinction of a partition, or, in cognate terms, an equivalence or not, i.e., identity or difference.

Logic of pairs of elements is logic of partitions (or equivalence relations). Pair of elements are distinct (in different blocks) or indistinct (in same block).

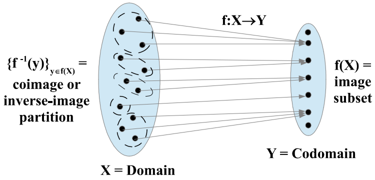
$$\left\{\ldots,\{\bullet,\ldots\},\ldots,\{*,\ldots\},\ldots\right\} \text{ or } \left\{\ldots,\{\ldots\},\ldots,\{\bullet,*,\ldots\},\ldots\right\}$$

$(\bullet,*)$ is a distinction

$(\bullet,*)$ is an indistinction, i.e. superposition at partition logic level

Logic of partitions–why so long?: I

- Boolean logic should be interpreted as logic of *subsets* of a universe U. Propositional logic is a special case of $U = 1$.
- Valid formula := formula that evaluates to universe set U for any subsets substituted for variables.
- *Truth table* validity should be theorem, not definition.
- Almost all logic texts only give propositional logic.
- Category theory duality gives subset-partition duality:



Logic of partitions–why so long?: II

- In category theory, subsets generalize to subobjects or "parts". *"The dual notion (obtained by reversing the arrows) of 'part' is the notion of partition."* (Lawvere)
- Propositions have no "dual" so the idea of a dual partition logic was not "in the air."
- Also Partition = Equivalence relation = Quotient set.
- *Lattice* of partitions known in 19th century (e.g., Dedekind & Schroder). Only partition ops. of join and meet.
- But *logic* needs operation of implication which was only defined in 21st century, e.g., DE, 2010. "The Logic of Partitions: Introduction to the Dual of the Logic of Subsets." *Review of Symbolic Logic* 3 (2 June): 287–350.

Logical algebras of subsets and partitions: I

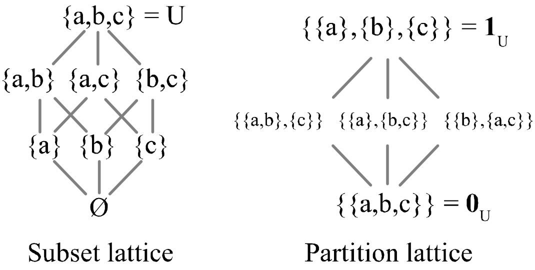
- Given universe set U, there is the *Boolean algebra of subsets* $\wp(U)$ with inclusion as partial ordering and the usual union and intersection, and enriched with implication or conditional: $S \supset T := S^c \cup T$ for $S, T \subseteq U$.
- A *partition* $\pi = \{B, ..., B'\}$ on U is a set of non-empty subsets of U that are disjoint and union is U.
- A *distinction* or *dit* of π is an ordered pair of elements of U in different blocks of π. The set of all dits of π is $\operatorname{dit}(\pi)$ and its complement in $U \times U - \operatorname{dit}(\pi) = \operatorname{indit}(\pi)$ is the associated equivalence relation.
- Given universe set U, there is the *algebra of partitions* $\Pi(U)$ with join and meet enriched by implication where refinement is the partial ordering.

Logical algebras of subsets and partitions: II

- Given partitions $\pi = \{B, ...\}$ and $\sigma = \{C, ...\}$, σ is *refined* by π, i.e., $\sigma \preceq \pi$, if for every block $B \in \pi$, there is a block $C \in \sigma$ such that $B \subseteq C$. Moreover, $\sigma \preceq \pi$ iff $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$.
- *Join* $\pi \vee \sigma$ is partition whose blocks are non-empty intersections $B \cap C$.
- *Meet* $\pi \wedge \sigma$: $\operatorname{indit}(\pi \wedge \sigma)$ is the reflexive-symmetric-transitive closure of $\operatorname{indit}(\pi) \cup \operatorname{indit}(\sigma)$.
- *Top* $\mathbf{1}_U = \{\{u\} : u \in U\}$ = *discrete* partition;
- *Bottom* $\mathbf{0}_U = \{U\}$ = *indiscrete* partition = "The Blob"
- *Implication* $\sigma \Rightarrow \pi$ is the partition that is like π except that any block $B \in \pi$ contained in some block $C \in \sigma$ is discretized. Discretized B like $\mathbf{1}_B$ & Undiscretized B like $\mathbf{0}_B$ so $\sigma \Rightarrow \pi$ is a characteristic function for refinement. Then

$$\sigma \preceq \pi \text{ iff } \sigma \Rightarrow \pi = \mathbf{1}_U$$
$$S \subseteq T \text{ iff } S \supset T = U.$$

Logical algebras of subsets and partitions: III



- The indiscrete partition $\mathbf{0}_U$ is called "The Blob" because like in the Hollywood movie of that name:

Logical algebras of subsets and partitions: IV

Absorption law for partition meet: $\mathbf{0}_U \wedge \pi = \mathbf{0}_U$
The Blob $\mathbf{0}_U$ absorbs everything it meets.

Tautologies in subset and partition logics: I

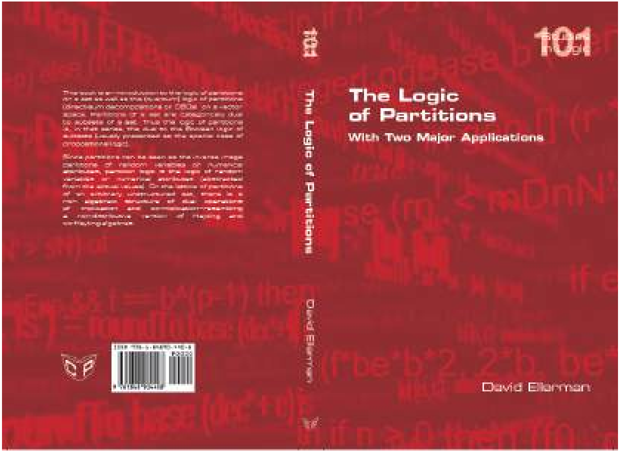
- A *subset tautology* is any formula which evaluates to U ($|U| \geq 1$) regardless of which subsets were assigned to the atomic variables.
- A *partition tautology* is any formula which always evaluates to $\mathbf{1}_U$ (the discrete partition) regardless of which partitions on U ($|U| \geq 2$) were assigned to the atomic variables.
- Every partition tautology is a subset tautology since $\Pi(2) \cong \wp(1)$.
- Partition tautologies neither included in nor include Intuitionistic tautologies (Heyting algebra validities). The weak law of excluded middle $\neg\sigma \vee \neg\neg\sigma$ is a partition but not an intuitionistic tautology and distributivity is a intuitionistic but not partition tautology.

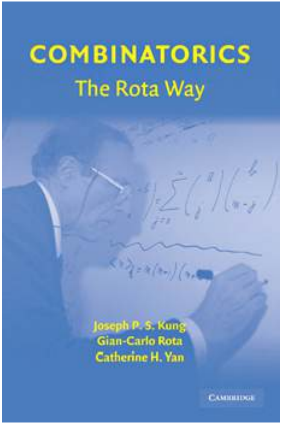
Tautologies in subset and partition logics: II

- In 1880, Charles Saunders Pierce published a paper claiming that he had proved all lattices distributive but the proof was too tedious to publish. Dedekind, Schroder, and others quickly disabused him with $\Pi(\{a,b,c\})$.
- Basic open questions in partition logic, i.e., "early days":
  - A decision procedure for partition tautologies.
  - A Hilbert-style axiom system for partition tautologies, plus a completeness proof for that axiom system.

Fund. Duality	Subset side "Its"	Partition side "Dits"
Logic ✓	*Logic of subsets*	*Logic of partitions*
Quant. Logic	Logical Probability	Logical Information
CT Duality	Subobj. & limits	Quot. Obj. & colimits
CT Canonicity	If $S \subseteq T$, $S \rightarrowtail T$	If $\sigma \precsim \pi$, $\pi \twoheadrightarrow \sigma$
Creation	Ex Nihilo creation	Big Bang creation
Physics	Fully def. Cl. Mech.	Indef. Q. Mech.

Tautologies in subset and partition logics: III





"The lattice of partitions plays for information the role that the Boolean algebra of subsets plays for size or probability."

$$\frac{\text{Subsets}}{\text{Probability}} \approx \frac{\text{Partitions}}{\text{Information}}$$

Quantitative Partition Logic: II

- Boole's quantitative logic of subsets = finite prob. theory = normalized number of elements in $S \subseteq U$, i.e., $\Pr(S) = \frac{|S|}{|U|}$.
- Rota, in his Fubini Lectures, said since "Probability is a measure on the Boolean algebra of events" that gives quantitatively the "intuitive idea of the size of a set", we may ask by "analogy" for some measure "which will capture some property that will turn out to be for [partitions] what size is to a set."
- Answer is *distinctions* or *dits* of a partition, i.e.,ordered pairs of elements in different blocks.

Quantitative Partition Logic: III

Lattice of subsets $\wp(U)$	Lattice of partitions $\Pi(U)$
Its = Elements of subsets	Dits = Distinctions of partitions
PO Incl. of subsets $S \subseteq T$	PO $\sigma \precsim \pi$ iff $\operatorname{dit}(\sigma) \subseteq \operatorname{dit}(\pi)$
Join: $S \vee T = S \cup T$	Join: $\operatorname{dit}(\sigma \vee \pi) = \operatorname{dit}(\sigma) \cup \operatorname{dit}(\pi)$
Top: U all elements	Top: $\mathbf{1}_U$ with all dits
Bottom: $\emptyset$ no elements	Bottom: $\mathbf{0}_U$ with no dits

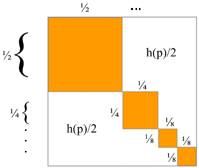
$$\frac{\text{Subsets}}{\text{Probability}} \approx \frac{\text{Partitions}}{\text{Information}} \text{ and } \frac{\text{Its}}{\text{Subsets}} \approx \frac{\text{Dits}}{\text{Partitions}}$$

- Then the definition of "logical information" or *logical entropy* is clear:

$$h(\pi) = \frac{|\operatorname{dit}(\pi)|}{|U \times U|} = \frac{|U \times U| - |\cup_j B_j \times B_j|}{|U \times U|} = 1 - \sum_j \left(\frac{|B_j|}{|U|}\right)^2 =$$

$$1 - \sum_j \Pr(B_j)^2 = \sum_{j \neq k} \Pr(B_j) \Pr(B_k).$$

- For a probability dist. on U, $p = (p_1, ..., p_n)$, $h(p) = 1 - \sum_i p_i^2 = \sum_{j \neq k} p_j p_k$.



Logical entropy box diagram for $p = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}\right)$ where $h(p) = \frac{21}{32}$.

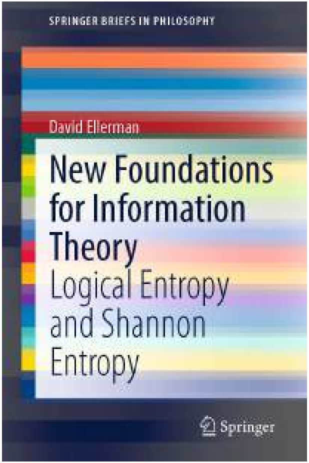
- $\Pr(S)$ = prob. of one draw from U getting an it of S.
- $h(\pi)$ = prob. of two draws from U getting a dit of π.
- $h(p)$ = prob. of two draws of different indices p_i and p_j.

Quantitative Partition Logic: V

- Logical information theory as the quantitative version of the logic of partitions provides a new *logical* foundation for information theory.

Fund. Duality	Subset side "Its"	Partition side "Dits"
Logic	Logic of subsets	Logic of partitions
Quant. Logic ✓	*Logical Probability*	*Logical Information*
CT Duality	Subobj. & limits	Quot. Obj. & colimits
CT Canonicity	If $S \subseteq T$, $S \rightarrowtail T$	If $\sigma \precsim \pi$, $\pi \twoheadrightarrow \sigma$
Creation	Ex Nihilo creation	Big Bang creation
Physics	Fully def. Cl. Mech.	Indef. Q. Mech.

Quantitative Partition Logic: VI



Its & Dits analysis of set functions in *Sets*: I

- A binary relation $R \subseteq X \times Y$ *preserves (or transmits) elements (Its)* if for each element $x \in X$, there is an ordered pair $(x, y) \in R$ for some $y \in Y$.
- A binary relation $R \subseteq X \times Y$ *reflects elements (Its)* if for each element $y \in Y$, there is an ordered pair $(x, y) \in R$ for some $x \in X$.
- A binary relation $R \subseteq X \times Y$ *preserves (or transmits) distinctions (Dits)* if for any pairs (x, y) and (x', y') in R, if $x \neq x'$, then $y \neq y'$.
- A binary relation $R \subseteq X \times Y$ *reflects distinctions (Dits)* if for any pairs (x, y) and (x', y') in R, if $y \neq y'$, then $x \neq x'$.
- A set function $f : X \rightarrow Y$ is usually characterized as being defined everywhere and single-valued, but:

Its & Dits analysis of set functions in *Sets*: II

- "Defined everywhere" is the same as "preserves elements" and
- "Being single-valued" is the same as "reflecting distinctions."

Binary relation is a *function*
iff it preserves elements and reflects distinctions.

- What about the other two notions of "preserving distinctions" and "reflecting elements"?
  - A function $f : X \to Y$ is *injective* iff it preserves distinctions; and
  - A function $f : X \to Y$ if *surjective* iff it reflects elements.
- Thus Its & Dits provide the natural language to define the morphisms in *Sets* and the special types of injections and surjections.

CT Duality = interchange Its & Dits I

- Defining a function $f \subseteq X \times Y$ as a relation "everywhere defined and single-valued" gives *no hint of duality*.
- Its & Dits-definition says just interchange Its & Dits like interchanging points and lines in plane proj. geometry.
- Interchange roles of Its & Dits in a function $f : X \to Y$ gives a cofunction $f^{op} : Y \to X$:

Function: A binary relation that preserves Its and reflects Dits.

Cofunction: A binary relation that preserves Dits and reflects Its.

- That is, a *concrete* morphism in $Sets^{op}$ is a binary relation, a *cofunction*, that preserves distinctions and reflects elements–instead of the opposite.

CT Duality = interchange Its & Dits II

- For the universal constructions in *Sets*, the interchange in the roles of elements (Its) and distinctions (Dits) interchanges each construction and its dual: products and coproducts, equalizers and coequalizers, and in general limits and colimits.
- That is then abstracted to make the reverse-the-arrows duality in abstract category theory.

The terminal object, initial object, and epi-mono factorization in *Sets*: I

- The bottom of the partition lattice, the indiscrete partition $\mathbf{0}_U = \{U\}$, is refined by all partitions on U, e.g., $\mathbf{0}_U \precsim \mathbf{1}_U$ so there is a p.o.-induced map $U \cong \mathbf{1}_U \to \mathbf{0}_U \cong 1$ ('the' one-element set). Taking U as any set in *Sets*, this canonical map $U \to 1$ (surjection if $U \neq \emptyset$) establishes 1 as the *terminal object* in *Sets*.
- The top of the Boolean lattice $\wp(U)$ is U, so each subset $S \subseteq U$ induces the canonical injection $S \to U$.
- The bottom of the subset lattice, the empty set $\emptyset$, is contained in every subset of U, e.g., $\emptyset \subseteq U$ so there is a canonical injection $\emptyset \to U$. Taking U as any set in *Sets*, this canonical injection establishes $\emptyset$ as the *initial object* in *Sets*.

The terminal object, initial object, and epi-mono factorization in *Sets*: II

	Subset logic	Partition logic
Partial order	$S \subseteq T$	$\sigma \precsim \pi$
Canonical map	$S \rightarrowtail T$	$\pi \twoheadrightarrow \sigma$
Extremal objects *Sets*	$\emptyset \subseteq U$ so $\emptyset \rightarrowtail U$	$\mathbf{0}_U \precsim \mathbf{1}_U$ so $U \twoheadrightarrow 1$

- Epi-Mono: Any function: $f : X \rightarrow Y$ gives the *coimage* $f^{-1} = \{f^{-1}(y) : y \in f(X)\}$ on X and the *image* $f(X) \subseteq Y$.
- Now $f^{-1} \precsim \mathbf{1}_X$, induces the canonical surjection: $X \cong \mathbf{1}_X \twoheadrightarrow f^{-1} \cong f(X)$.
- And $f(X) \subseteq Y$ induces the canonical injection $f(X) \rightarrowtail Y$.
- The epi-mono factorization of f is the composition of the canonical maps: $f : X \rightarrow Y = X \twoheadrightarrow f(X) \rightarrowtail Y$.

Its & Dits analysis of coequalizers: I

- For the equalizer and coequalizer, the data is not just two sets but two parallel maps $f, g : X \rightrightarrows Y$.
- Then each element $x \in X$, gives us a pair $f(x)$ and $g(x)$ so we take the equivalence relation $\sim$ defined on Y that is generated by $f(x) \sim g(x)$ for any $x \in X$.
- Then the coequalizer is the quotient set $C = Y/\sim$.
- When $\sim$ is represented as a partition on Y, then it is refined by the discrete partition $\mathbf{1}_Y$ on Y, and that refinement defines the canonical surjection $Y \cong \mathbf{1}_Y \to Y/\sim$.
- For the UMP, let $h : Y \to Z$ be such that $hf = hg$. Then we need to show there is a unique refinement-defined map $h^* : Y/\sim \to Z$ such that the triangle commutes.

Its & Dits analysis of coequalizers: II

$$
\begin{array}{ccccc}
X & \overset{f}{\underset{g}{\rightrightarrows}} & Y & \twoheadrightarrow & Y/\sim \\
& & & \searrow^{h} \quad \exists! & \downarrow h^{*} \\
& & & & Z
\end{array}
$$

Coequalizer diagram

- We already have one partition $\sim$ on Y which was generated by $f(x) \sim g(x)$.
- Since $hf = hg$, we are given that $hf(x) = hg(x)$ so the coimage h^{-1} has to *at least* identify $f(x)$ and $g(x)$ (and perhaps identify other elements) which means that $h^{-1} \precsim Y/\sim$ in the partition lattice on Y.

Its & Dits analysis of coequalizers: III

- Hence the induced surjection map $Y/\sim \twoheadrightarrow h^{-1}$ and the mono $h^{-1} \cong h(Y) \rightarrowtail Z$ (taking $h^{-1}(z)$ to z) completes the factor map $h^{*} : Y/\tilde{} \twoheadrightarrow h^{-1} \rightarrowtail Z$.

Fund. Duality	Subset side "Its"	Partition side "Dits"
Logic	Logic of subsets	Logic of partitions
Quant. Logic	Logical Probability	Logical Information
CT Duality ✓	*Subobj. & limits*	*Quot. Obj. & colimits*
CT Canonicity	If $S \subseteq T$, $S \rightarrowtail T$	If $\sigma \precsim \pi$, $\pi \twoheadrightarrow \sigma$
Creation	Ex Nihilo creation	Big Bang creation
Physics	Fully def. Cl. Mech.	Indef. Q. Mech.

- The *logical theory of canonicity*, based on Its & Dits duality, characterizes canonical maps as arising (using the given data) from the partial orders in the two dual logics of subsets and of partitions:
  - Boolean lattice $\wp(U)$ of subsets of U: PO is inclusion $S \subseteq T$ which induces the canonical injection: $S \rightarrowtail T$;
  - Partition lattice $\Pi(U)$ of partitions on a non-empty U: PO is refinement $\sigma \precsim \pi$ for $\sigma, \pi \in \Pi(U)$, which means for every block $B \in \pi$, there is a block $C \in \sigma$ such that $B \subseteq C$ and which induces the canonical surjection: $\pi \twoheadrightarrow \sigma$.

Its & Dits analysis of *canonical* maps: II

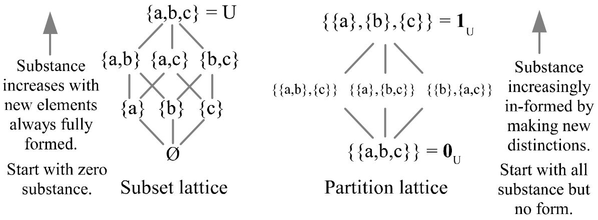
Canonical maps = maps induced from given data by partial orders of the two logics of subsets and partitions.

Fund. Duality	Subset side "Its"	Partition side "Dits"
Logic	Logic of subsets	Logic of partitions
Quant. Logic	Logical Probability	Logical Information
CT Duality	Subobj. & limits	Quot. Obj. & colimits
CT Canonicity ✓	*If* $S \subseteq T$, $S \rightarrowtail T$	*If* $\sigma \precsim \pi$, $\pi \twoheadrightarrow \sigma$
Creation	Ex Nihilo creation	Big Bang creation
Physics	Fully def. Cl. Mech.	Indef. Q. Mech.

Its & Dits Duality: Two ways to combine Substance and Form: I

- We shown how the dual concepts of Its & Dits can be used to account for the notion of morphism and duality in *Sets*–which are then abstracted in abstract category theory.
- Hence the Its & Dits notions may have a broader philosophical significance.
- One possibility is they are respectively mathematical versions of the old metaphysical concepts of *matter* (or *substance*) and *form* (as in in-form-ation).

Its & Dits Duality: Two ways to combine Substance and Form: II



- At the bottom of the Boolean subset lattice is the empty set $\emptyset$ which represents no substance or matter (no 'its').
- At the bottom of the partition lattice is the indiscrete partition or "blob" $\mathbf{0} = \{U\}$ (where the universe set U makes one block) which represents all the substance or matter but with no distinctions to in-form the substance (no 'dits').

Its & Dits Duality: Two ways to combine Substance and Form: III

- Thus, moving upwards, one ends up at the "same place" (universe U of fully distinguished elements) either way, but by two totally different but dual ‘creation stories’:
  - creating elements (as in *ex nihilo*) versus
  - creating distinctions (from undifferentiated matter and then breaking symmetries as in Big Bang).

Fund.l Duality	Subset side “Its”	Partition side “Dits”
Logic	Logic of subsets	Logic of partitions
Quant. Logic	Logical Probability	Logical Information
CT Duality	Subobj. & limits	Quot. Obj. & colimits
CT Canonicity	If $S \subseteq T$, $S \rightarrowtail T$	If $\sigma \precsim \pi$, $\pi \twoheadrightarrow \sigma$
Creation ✓	*Ex Nihilo creation*	*Big Bang creation*
Physics	Fully def. Cl. Mech.	Indef. Q. Mech.

The metaphysics of classical physics = Definite-world:

- The relevant characteristic of classical physics (and common-sense reality) is "definite all the way down."
- Leibniz's Principle of Identity of Indiscernibles:

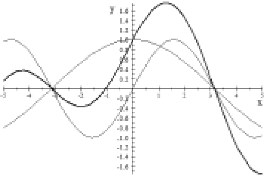
  If two entities are distinct, then there is always some property applying to one but not the other to distinguish them. Hence if indistinguishable, then they are same thing.

- Kant's Principle of Complete Determination:

  Every thing, however, as to its possibility, further stands under the principle of thoroughgoing determination; according to which, among all possible predicates of things, insofar as they are compared with their opposites, one must apply to it.

QM = Indefinite-world & not Wave-world: I

- The key non-classical notion is superposition (entanglement is a special case).
- A superposition state is *not* simultaneously in two definite states but is *objectively indefinite* between the different definite (eigen-) states.
- The most misleading imagery for superposition was the sum of two definite waves giving another *definite* wave.



QM = Indefinite-world & not Wave-world: II

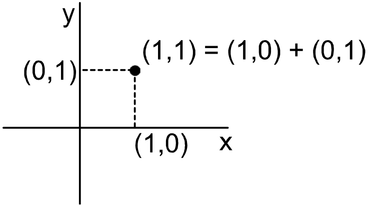
Such analogies have led to the name 'Wave Mechanics' being sometimes given to quantum mechanics. It is important to remember, however, that the superposition that occurs in quantum mechanics is of an essentially different nature from any occurring in the classical theory, as is shown by the fact that the ***quantum superposition principle demands indeterminacy*** *in the results of observations in order to be capable of a sensible physical interpretation. The analogies are thus liable to be misleading. (Dirac, Principles)*

- The math is not wrong since complex numbers are the natural math for waves (i.e., polar rep. is amplitude & phase); the imagery is wrong as an ontology.

QM = Indefinite-world & not Wave-world: III

- The math of QM (among other reasons) needs $\mathbb{C}$ since $\mathbb{C}$ is the algebraically-complete extension of the reals so the real-valued observables will have a full set of eigenvectors.
- QM math's interpretation as "wave mechanics" is only a (misleading) artifact of using Hilbert space over $\mathbb{C}$.
- The **real point** is that QM math is the Hilbert space version of the math of partitions (i.e., indefiniteness & definiteness). QM math is better thought of as the physics of particles in indefinite states than "wave mechanics".

QM = Indefinite-world & not Wave-world: IV

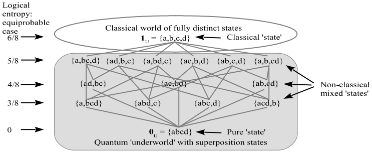


- The state $(1,1)$ is indefinite (with equal amplitudes) betweens the state of definite x-ness and definite y-ness.

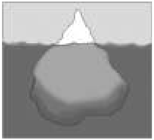
Partitions: The math of indefiniteness: I

- The (Hasse) diagram for a lattice of partitions represents in skeletal form (i.e., no scalars) the *support set* (non-zero eigenstates in a superposition) for a particle:
  - particle state $\alpha |a\rangle + \beta |b\rangle + \gamma |c\rangle$ skeletonizes to its non-zero support set as a partition block $\{a, b, c\}$,
  - Singleton block = classical state,
  - Non-singleton block = superposition state, e.g., $\{a, b, c\}$ or abc (shorthand) is particle in superposition of eigenstates a, b, c.
  - Mixture of singleton & non-singleton blocks = mixed state.

Partitions: The math of indefiniteness: II



Lattice of Supports



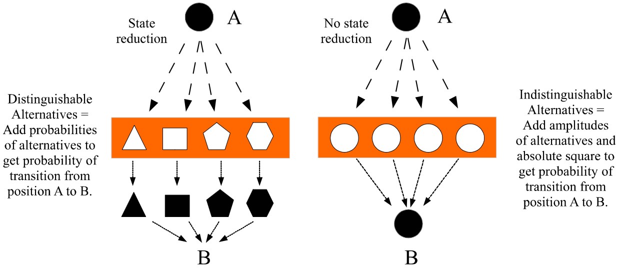
- Quantum underworld = Heisenberg/Shimony/Margenau/Hughes/et al. real potentia/latencies.
- Top = discrete partition $\mathbf{1}_U = \{\{u_1\}, ..., \{u_n\}\}$ of only singleton blocks (so only self-pairs $(u_i, u_i) \in \Delta$ are indits) which represents classical reality (no superpositions) so the criterion of classicality is:

$$\text{For all } u, u' \in U, \text{ if } (u, u') \in \text{indit}(\mathbf{1}_U), \text{ then } u = u'.$$

Law of Identity of Indistinguishables (partition version)

Closing the circle: State reduction = inverse of superposition

- Interpreting superposition as making eigenstates indistinct is key to see that **state reduction is the inverse process**, i.e., an interaction that makes superposed eigenstates distinct or distinguished, i.e., Feynman Rule.



Feynman Rule: for blob going thru (Weyl's) grating or pasta machine

Logical entropy measures state reduction: I

- Now it all ties together:
  - Superposition is indistinct between eigenstates,
  - State reduction = interaction that makes distinctions between eigenstates,
  - logical entropy quantifies distinctions so it measures state reduction (upward change in iceberg diagram).
  - This is why so many quantum theorists have sensed a connection between QM & information.
- Classical reality emerges from quantum reality by making of distinctions.
- *Lüders mixture operation* describes changes in density matrix with PVM: $\hat{\rho} = \sum_{r \in f(U)} P_r \rho P_r$.
- Non-zero off-diagonal elements ρ_{jk} = quantum coherences = quantum indistinctions of ρ.

Logical entropy measures state reduction: II

- Quantum indistinctions zeroed $\rho_{jk} \rightsquigarrow 0$ in $\rho \rightsquigarrow \hat{\rho}$ are *new* quantum distinctions.

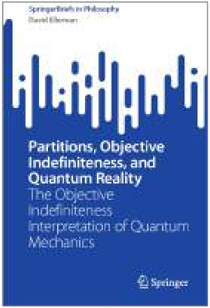
Theorem

$h(\hat{\rho}) - h(\rho) = \sum_{\rho_{jk} \rightsquigarrow 0} |\rho_{jk}|^2$. *Lüders mixture operation*

- This is how "Logical entropy measures 'measurement' "–so it all fits together precisely where $h(\rho) = 1 - \mathrm{tr}[\rho^2]$.

Fund. Duality	Subset side "Its"	Partition side "Dits"
Logic	Logic of subsets	Logic of partitions
Quant. Logic	Logical Probability	Logical Information
CT Duality	Subobj. & limits	Quot. Obj. & colimits
CT Canonicity	If $S \subseteq T$, $S \rightarrowtail T$	If $\sigma \precsim \pi$, $\pi \twoheadrightarrow \sigma$
Creation	Ex Nihilo creation	Big Bang creation
Physics ✓	*Fully def. Cl. Mech.*	*Indef. Q. Mechanics*

Logical entropy measures state reduction: III



(free in all the usual hacker book sites: Anna's Archive, Z-library, or in Zagreb: www.memoryoftheworld.org)

Website: www.ellerman.org and

Substack: https://davidellerman.substack.com/

Thank you