Tense Logics, Structural Proof Theory, and Effective Translations

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LOGICAL METHODS IN COMPUTER SCIENCE

TICAMORE: Translating and dIscovering CAlculi for MOdal and REIated logics

$Rxy, Rxz, x : A, y : B, y : C$

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$A, B \vdash C, D, E$

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$A \vdash B | C, D \vdash E | \vdash F$

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Et cetera . . .

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Et cetera . . .

Tense Logic: Introduction

- A logic for reasoning about logical notions of time.
- Language (Negation Normal Form):

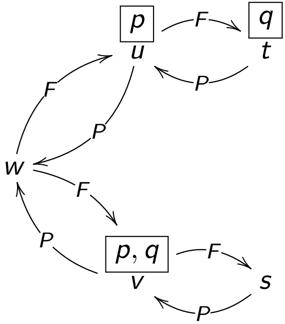
$$A := p \mid \overline{p} \mid A \lor A \mid A \land A \mid \square A \mid \blacksquare A \mid \lozenge A \mid \blacklozenge A$$

*We take implication $\rightarrow$, negation $\neg$, and bi-implication $\leftrightarrow$ to be defined.

- Interpretations:
  - $\square A$ - "A holds at every point in the future."
  - $\blacksquare A$ - "A holds at every point in the past."
  - $\lozenge A$ - "A holds at some point in the future."
  - $\blacklozenge A$ - "A holds at some point in the past."

Semantics: Tense Kripke Models

- A Tense Kripke Model is a tuple (W, R_F, R_P, V):
  - W is a set of points
  - $R_F \subseteq W \times W$
  - $R_P = \{(w,u) | (u,w) \in R_F\}$
  - $V : Prop \to 2^W$
- Example:
  - $W = \{w, u, v, t, s\}$
  - $R_F = \{(w,u), (w,v), (u,t), (v,s)\}$
  - $R_P = \{(u,w), (v,w), (t,u), (s,v)\}$
  - $V(p) = \{u, v\}$
  - $V(q) = \{t, v\}$



Semantics: Satisfaction, Global Truth, Validity

Definition (Satisfaction)

$\mathfrak{M}$ is a Tense Kripke Model, with w a point in the model:

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \overline{p}$ iff $w \notin V(p)$
- $\mathfrak{M}, w \models A \vee B$ iff $\mathfrak{M}, w \models A$ or $\mathfrak{M}, w \models B$
- $\mathfrak{M}, w \models A \wedge B$ iff $\mathfrak{M}, w \models A$ and $\mathfrak{M}, w \models B$
- $\mathfrak{M}, w \models \Box A$ iff $\forall u$ if $R_F wu$, then $\mathfrak{M}, u \models A$
- $\mathfrak{M}, w \models \Diamond A$ iff $\exists u$ $R_F wu$ and $\mathfrak{M}, u \models A$
- $\mathfrak{M}, w \models \blacksquare A$ iff $\forall u$ if $R_P wu$, then $\mathfrak{M}, u \models A$
- $\mathfrak{M}, w \models \blacklozenge A$ iff $\exists u$ $R_P wu$ and $\mathfrak{M}, u \models A$

- *Globally truth* $\equiv$ holds at every point of the model.
- *Validity* $\equiv$ globally true on every model.

Tense Logic: Hilbert Style Axiomatization

Hilbert Calculus:

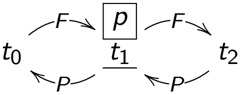
Axioms:	Inference Rules:
$A \to (B \to A)$	$\dfrac{A \qquad A \to B}{B}$ (MP)
$A \to (B \to C) \to ((A \to B) \to (A \to C))$	$\dfrac{A}{\Box A}$ ($\Box$)
$(\neg B \to \neg A) \to (A \to B)$	$\dfrac{A}{\blacksquare A}$ ($\blacksquare$)
$\Box(A \to B) \to (\Box A \to \Box B)$	
$\Box A \leftrightarrow \neg\Diamond\neg A$	
$\blacksquare(A \to B) \to (\blacksquare A \to \blacksquare B)$	
$\blacksquare A \leftrightarrow \neg\blacklozenge\neg A$	
$A \to \Box\blacklozenge A$	
$A \to \blacksquare\Diamond A$	

Definition

The Minimal Tense Logic Kt

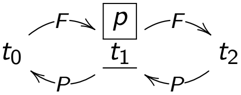
Interpretations and Extensions

- Converse Axioms $p \rightarrow \square \blacklozenge p$ and $p \rightarrow \blacksquare \lozenge p$:

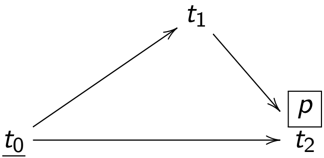


Interpretations and Extensions

- Converse Axioms $p \to \Box\blacklozenge p$ and $p \to \blacksquare\Diamond p$:



- Transitivity $\Diamond\Diamond p \to \Diamond p$:



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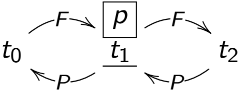

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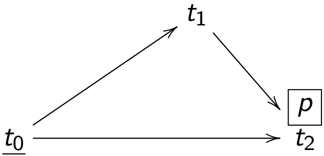
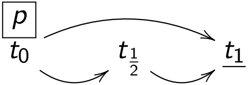


- Density $\blacklozenge p \to \blacklozenge\blacklozenge p$:



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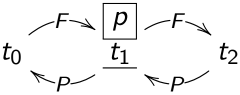

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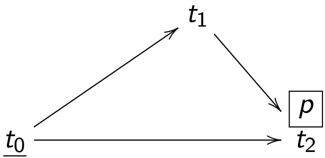

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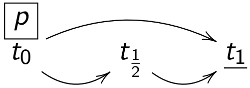

Definition

General Path Axioms: $\Pi p \to \Sigma p$ for $\Pi, \Sigma \in \{\Diamond, \blacklozenge\}^*$

A Display Calculus for Tense Logic

Language:

$$X := A \mid X, X \mid \circ\{X\} \mid \bullet\{X\}$$

where A is a tense logic formula.

Definition (Goré *et al.* 2011)

The Display Calculus SKT:

$$\frac{}{\Gamma, p, \overline{p}}\ (id) \qquad \frac{\Gamma, A, B}{\Gamma, A \vee B}\ (\vee) \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \wedge B}\ (\wedge) \qquad \frac{\Gamma, A \qquad \Gamma, \overline{A}}{\Gamma}\ (Cut)$$

$$\frac{\Gamma, \Delta, \Delta}{\Gamma, \Delta}\ (ctr) \qquad \frac{\Gamma}{\Gamma, \Delta}\ (wk) \qquad \frac{\Gamma, \circ\{\Delta\}}{\bullet\{\Gamma\}, \Delta}\ (rf) \qquad \frac{\Gamma, \bullet\{\Delta\}}{\circ\{\Gamma\}, \Delta}\ (rp)$$

$$\frac{\Gamma, \bullet\{A\}}{\Gamma, \blacksquare A}\ (\blacksquare) \qquad \frac{\Gamma, \circ\{A\}}{\Gamma, \Box A}\ (\Box) \qquad \frac{\Gamma, \bullet\{\Delta, A\}, \blacklozenge A}{\Gamma, \bullet\{\Delta\}, \blacklozenge A}\ (\blacklozenge) \qquad \frac{\Gamma, \circ\{\Delta, A\}, \Diamond A}{\Gamma, \circ\{\Delta\}, \Diamond A}\ (\Diamond)$$

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$$\frac{\Gamma, \Delta, \Delta}{\Gamma, \Delta}\ (\text{ctr}) \quad \frac{\Gamma}{\Gamma, \Delta}\ (\text{wk}) \quad \frac{\Gamma, \circ\{\Delta\}}{\bullet\{\Gamma\}, \Delta}\ (\text{rf}) \quad \frac{\Gamma, \bullet\{\Delta\}}{\circ\{\Gamma\}, \Delta}\ (\text{rp})$$

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Example: Display Proof

$$\frac{}{\Gamma, p, \overline{p}}\ (id) \qquad \frac{\Gamma, A, B}{\Gamma, A \lor B}\ (\lor) \qquad \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \land B}\ (\land)$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{q, \overline{q}, t, \circ\{r\} \qquad q, \overline{q}, s, \circ\{r\}}{q, \overline{q}, t \land s, \circ\{r\}}\ (\land)}{(q \lor \overline{q}), t \land s, \circ\{r\}}\ (\lor)}{(q \lor \overline{q}) \lor (t \land s), \circ\{r\}}\ (\lor)}{\bullet\{(q \lor \overline{q}) \lor (t \land s)\}, r}\ (rf)}{\blacksquare((q \lor \overline{q}) \lor (t \land s)), r}\ (\blacksquare)}{\blacksquare((q \lor \overline{q}) \lor (t \land s)) \lor r}\ (\lor)$$

Cut-free Extensions

- General Path Axioms: $\Pi p \to \Sigma p \quad (\Pi, \Sigma \in \{\Diamond, \blacklozenge\}^*)$

$$\frac{\Gamma, \star_1\{\dots \star_m \{\Delta\}\dots\}}{\Gamma, *_1\{\dots *_n \{\Delta\}\dots\}}\, GP \quad \equiv \quad \Pi p \to \Sigma p \quad (\Pi, \Sigma \in \{\Diamond, \blacklozenge\}^*)$$

$\star_i = \circ$ for $\Diamond \in \Sigma$ and $\star_i = \bullet$ for $\blacklozenge \in \Sigma$.
$*_j = \circ$ for $\Diamond \in \Pi$ and $\star_i = \bullet$ for $\blacklozenge \in \Pi$.

Example

$$\Diamond\blacklozenge p \to \blacklozenge\Diamond\blacklozenge p \quad \equiv \quad \Box\blacksquare\overline{p} \vee \blacklozenge\Diamond\blacklozenge p \qquad \frac{\Gamma, \bullet\{\circ\{\bullet\{\Delta\}\}\}}{\Gamma, \circ\{\bullet\{\Delta\}\}}$$

- Structural rule $\approx$ corresponding axiom
- Preserves cut-admissibility

1. Motivations
2. Tense Logic
3. Display Calculi
4. **Labelled Calculi**
5. Translations
6. Applications & Future Work

A Labelled Calculus for Tense Logic

Language:

$$X := x : A \mid X, X \mid Rxy, X$$

where A is a tense logic formula.

Definition (Negri 2005)

The labelled sequent calculus G3Kt:

$$\frac{}{\mathcal{R}, x : p, x : \overline{p}, \Gamma} \text{(id)}$$

$$\frac{\mathcal{R}, x : A, x : B, \Gamma}{\mathcal{R}, x : A \vee B, \Gamma} (\vee) \qquad \frac{\mathcal{R}, x : A, \Gamma \qquad \mathcal{R}, x : B, \Gamma}{\mathcal{R}, x : A \wedge B, \Gamma} (\wedge)$$

$$\frac{\mathcal{R}, Rxy, y : A, \Gamma}{\mathcal{R}, x : \Box A, \Gamma} (\Box)^* \qquad \frac{\mathcal{R}, Ryx, y : A, \Gamma}{\mathcal{R}, x : \blacksquare A, \Gamma} (\blacksquare)^*$$

$$\frac{\mathcal{R}, Rxy, x : \Diamond A, y : A, \Gamma}{\mathcal{R}, Rxy, x : \Diamond A, \Gamma} (\Diamond) \qquad \frac{\mathcal{R}, Ryx, x : \blacklozenge A, y : A, \Gamma}{\mathcal{R}, Ryx, x : \blacklozenge A, \Gamma} (\blacklozenge)$$

*The variable y may not occur in the conclusion of $(\Box)$ or $(\blacksquare)$.

Example: Labelled Proof

$$\frac{}{\mathcal{R}, x : p, x : \overline{p}, \Gamma} \text{(id)} \qquad \frac{\mathcal{R}, x : A, \Gamma \qquad \mathcal{R}, x : B, \Gamma}{\mathcal{R}, x : A \land B, \Gamma} (\land)$$

$$\frac{\mathcal{R}, x : A, x : B, \Gamma}{\mathcal{R}, x : A \lor B, \Gamma} (\lor) \qquad \frac{\mathcal{R}, Ryx, y : A, \Gamma}{\mathcal{R}, x : \blacksquare A, \Gamma} (\blacksquare)^* \qquad \frac{\mathcal{R}, Rxy, y : A, \Gamma}{\mathcal{R}, x : \Box A, \Gamma} (\Box)^*$$

$$\frac{\mathcal{R}, Rxy, y : A, x : \Diamond A, \Gamma}{\mathcal{R}, Rxy, x : \Diamond A, \Gamma} (\Diamond) \qquad \frac{\mathcal{R}, Ryx, y : A, x : \blacklozenge A, \Gamma}{\mathcal{R}, Ryx, x : \blacklozenge A, \Gamma} (\blacklozenge)$$

Example

$$\cfrac{\cfrac{\cfrac{Rxy, x : \overline{p}, y : \blacklozenge p, x : p}{Rxy, x : \overline{p}, y : \blacklozenge p} (\blacklozenge)}{x : \overline{p}, x : \Box \blacklozenge p} (\Box)}{x : \overline{p} \lor \Box \blacklozenge p} (\lor)$$

Nice Properties: Contraction Admissibility

$$\dfrac{\mathcal{R}, Rxy, Rxy, \Gamma}{\mathcal{R}, Rxy, \Gamma}\ (\mathrm{Ctr}) \qquad \dfrac{\mathcal{R}, x : A, x : A, \Gamma}{\mathcal{R}, x : A, \Gamma}\ (\mathrm{Ctr})$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{Rxy, x : \overline{p}, y : \blacklozenge p, y : \blacklozenge p, x : p}{Rxy, x : \overline{p}, y : \blacklozenge p, y : \blacklozenge p}\ (\blacklozenge)}{Rxy, x : \overline{p}, y : \blacklozenge p}\ (\mathrm{Ctr})}{x : \overline{p}, x : \Box\blacklozenge p}}{x : \overline{p} \lor \Box\blacklozenge p} \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{Rxy, x : \overline{p}, y : \blacklozenge p, y : \blacklozenge p, x : p}{Rxy, x : \overline{p}, y : \blacklozenge p, x : p}\ (\mathrm{Ctr})}{Rxy, x : \overline{p}, y : \blacklozenge p}\ (\blacklozenge)}{x : \overline{p}, x : \Box\blacklozenge p}}{x : \overline{p} \lor \Box\blacklozenge p}$$

$$\Longrightarrow \quad \dfrac{\dfrac{\dfrac{Rxy, x : \overline{p}, y : \blacklozenge p, x : p}{Rxy, x : \overline{p}, y : \blacklozenge p}\ (\blacklozenge)}{x : \overline{p}, x : \Box\blacklozenge p}}{x : \overline{p} \lor \Box\blacklozenge p}$$

Nice Properties: Weakening Admissibility

$$\frac{\mathcal{R}, \Gamma}{\mathcal{R}, Rxy, \Gamma}\ (\mathrm{Wk}) \qquad\qquad \frac{\mathcal{R}, \Gamma}{\mathcal{R}, x : A, \Gamma}\ (\mathrm{Wk})$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Rxy, x : \overline{p}, y : \blacklozenge p, x : p}{Rxy, x : \overline{p}, y : \blacklozenge p}\ (\blacklozenge)}{Rxz, Rxy, x : \overline{p}, y : \blacklozenge p}\ (\mathrm{Wk})}{Rxz, x : \overline{p}, x : \Box\blacklozenge p}}{Rxz, x : \overline{p} \vee \Box\blacklozenge p}}{z : \blacksquare(\overline{p} \vee \Box\blacklozenge p)} \implies \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Rxy, x : \overline{p}, y : \blacklozenge p, x : p}{Rxz, Rxy, x : \overline{p}, y : \blacklozenge p, x : p}\ (\mathrm{Wk})}{Rxz, Rxy, x : \overline{p}, y : \blacklozenge p}\ (\blacklozenge)}{Rxz, x : \overline{p}, x : \Box\blacklozenge p}}{Rxz, x : \overline{p} \vee \Box\blacklozenge p}}{z : \blacksquare(\overline{p} \vee \Box\blacklozenge p)}$$

$$\implies \dfrac{\dfrac{\dfrac{\dfrac{Rxz, Rxy, x : \overline{p}, y : \blacklozenge p, x : p}{Rxz, Rxy, x : \overline{p}, y : \blacklozenge p}\ (\blacklozenge)}{Rxz, x : \overline{p}, x : \Box\blacklozenge p}}{Rxz, x : \overline{p} \vee \Box\blacklozenge p}}{z : \blacksquare(\overline{p} \vee \Box\blacklozenge p)}$$

Nice Properties: Cut Admissibility

$$\frac{\mathcal{R}, \Gamma, x : A \qquad \mathcal{R}, \Gamma, x : \overline{A}}{\mathcal{R}, \Gamma} \text{ (Cut)}$$

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{Rxy, x : \overline{p}, y : \blacklozenge p, x : p, x : \overline{p} \qquad Rxy, x : \overline{p}, y : \blacklozenge p, x : p, x : p}{Rxy, x : \overline{p}, y : \blacklozenge p, x : p} \text{ (Cut)}}{Rxy, x : \overline{p}, y : \blacklozenge p}}{x : \overline{p}, x : \Box \blacklozenge p}}{x : \overline{p} \vee \Box \blacklozenge p}$$

$$\cfrac{\cfrac{\cfrac{Rxy, x : \overline{p}, y : \blacklozenge p, x : p}{Rxy, x : \overline{p}, y : \blacklozenge p}}{x : \overline{p}, x : \Box \blacklozenge p}}{x : \overline{p} \vee \Box \blacklozenge p}$$

Cut-free Extensions

- General Path Axioms: $\Pi p \to \Sigma p \quad (\Pi, \Sigma \in \{\Diamond, \blacklozenge\}^*)$

- Can be transformed into equivalent rules:

$$\frac{\mathcal{R}, R_{\Pi}xy, R_{\Sigma}xy, \Gamma}{\mathcal{R}, R_{\Pi}xy, \Gamma}\ GP \quad \equiv \quad \Pi p \to \Sigma p \quad (\Pi, \Sigma \in \{\Diamond, \blacklozenge\}^*)$$

where $R_{\Diamond}xy = Rxy$ and $R_{\blacklozenge}xy = Ryx$;

$R_{\Pi}xy = R_{\langle ? \rangle_1}xy_1, \ldots, R_{\langle ? \rangle_m}y_m y$ for $\Pi = \langle ? \rangle_1 \ldots \langle ? \rangle_m$;

$R_{\Sigma}xy = R_{\langle ? \rangle_n}xz_1, \ldots, R_{\langle ? \rangle_n}y_n y$ for $\Sigma = \langle ? \rangle_1 \ldots \langle ? \rangle_n$.

- Preserves cut-admissibility

1. Motivations
2. Tense Logic
3. Display Calculi
4. Labelled Calculi
5. Translations
6. Applications & Future Work

Roadmap

1. Display Sequent and Labelled Sequent Graphs
2. Labelled Polytree Sequents
3. Translating Notation: Display to Labelled
4. Translating Notation: Labelled to Display
5. Example: Translating a Display Proof to a Labelled Proof
6. The Reverse Direction: From Labelled to Display

Visualizing Display Sequents

$$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$$

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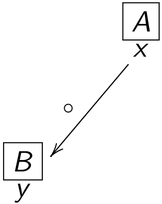
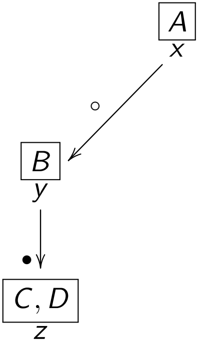


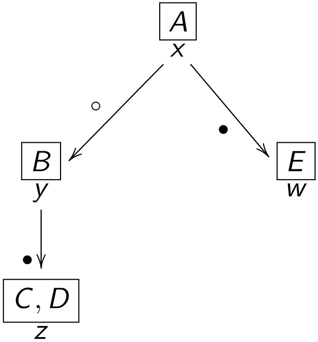
Visualizing Display Sequents

$$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$$



Visualizing Display Sequents

$$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$$



Visualizing Labelled Sequents

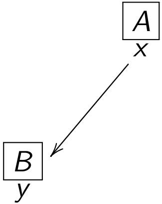
$$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$$

Visualizing Labelled Sequents

$$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$$

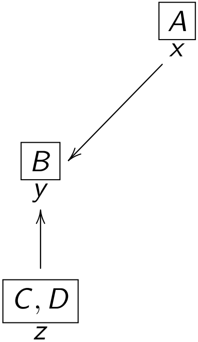
Visualizing Labelled Sequents

$$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$$



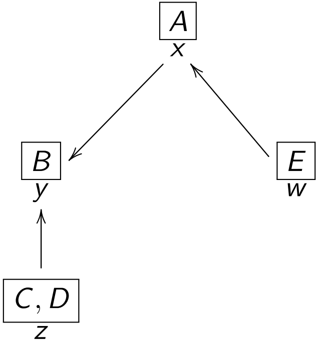
Visualizing Labelled Sequents

$$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$$



Visualizing Labelled Sequents

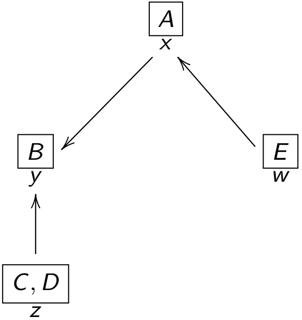
$$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$$



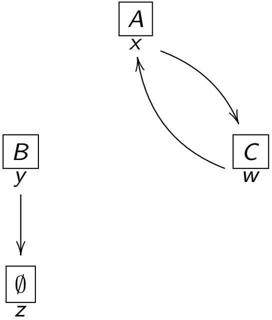
Which Labelled Sequents are Essentially Display?

- Labelled Polytree Sequent := sequent whose graph is a labelled polytree.
- Labelled Polytree := connected and acyclic directed graph.

$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$



$Rxw, Rwx, Ryz, x : A, y : B, w : C$



From Display to Labelled

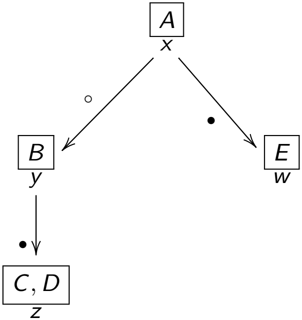
From Display to Labelled

- How to translate from display to labelled?

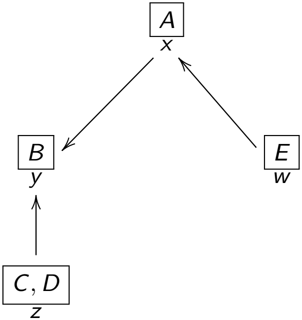
From Display to Labelled

- How to translate from display to labelled? Easy!
  - (1) Flip • edges and switch type
  - (2) Remove edge-typing

$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$



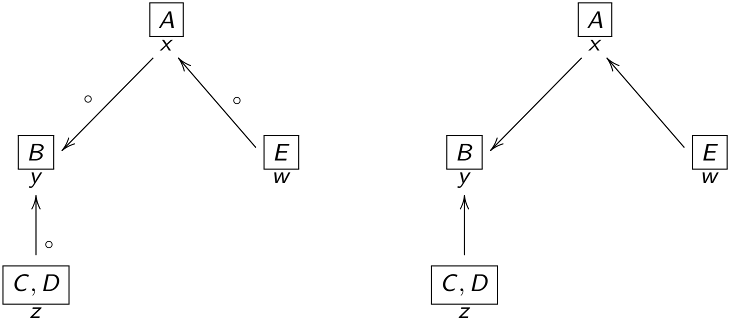
$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$



From Display to Labelled

- How to translate from display to labelled? Easy!
  - (1) Flip $\bullet$ edges and switch type
  - (2) Remove edge-typing

$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$ $\qquad$ $Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$



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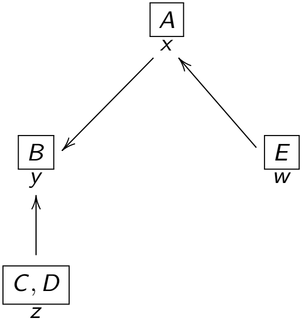
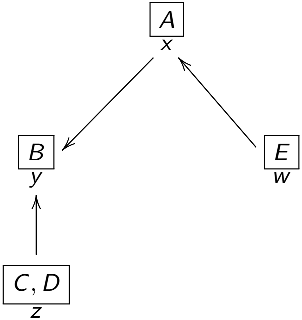
$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$





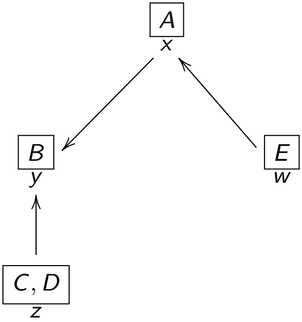
From Labelled to Display

- How to translate from labelled to display?

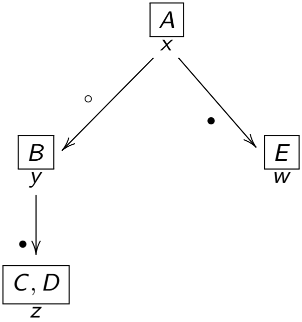
From Labelled to Display

- How to translate from labelled to display? Almost as Easy!
  - (1) Pick a node
  - (2) Moving through the tree from that node label forward edges with a $\circ$ and backward edges with a $\bullet$
  - (3) Reverse all $\bullet$ edges

$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$



$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$



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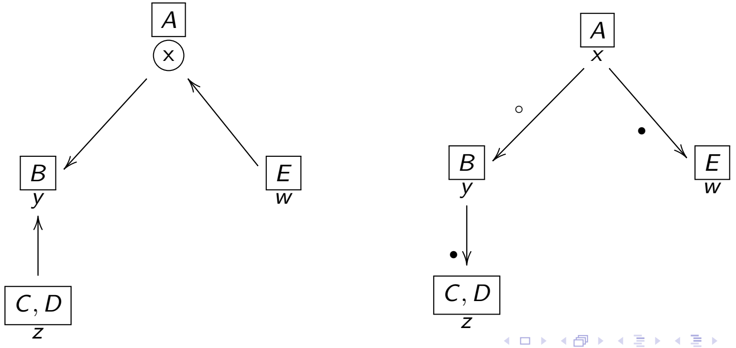
$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$

$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$



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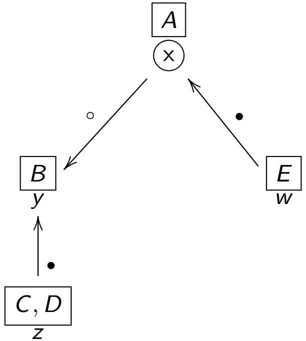
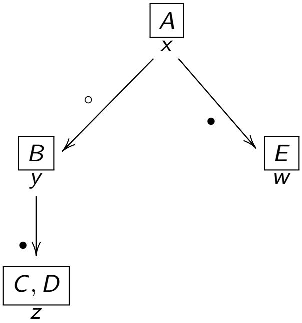


$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$



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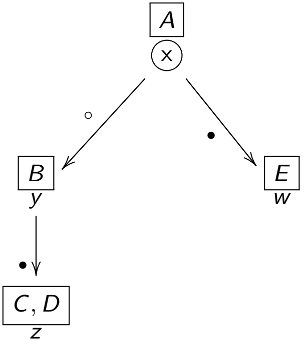
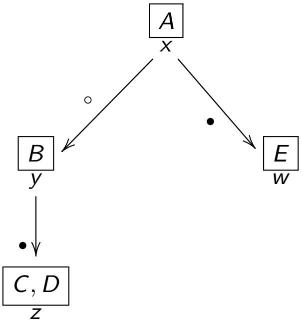
$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$



$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$



From Labelled to Display–A Problem?

- How to translate from labelled to display? Almost as Easy!
  - (1) Pick a node
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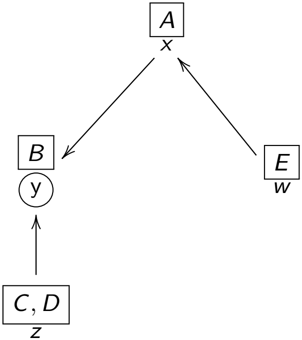
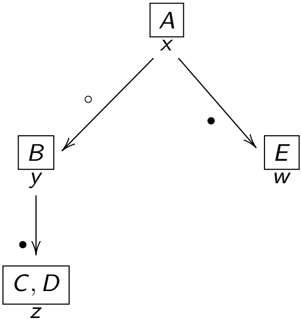
$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$



$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$



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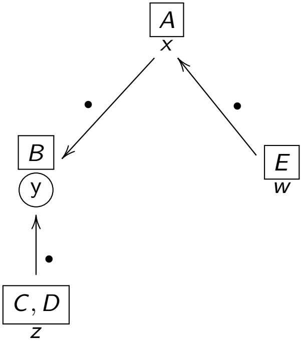
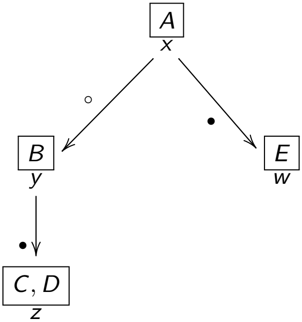
$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$



$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$



From Labelled to Display–A Problem?

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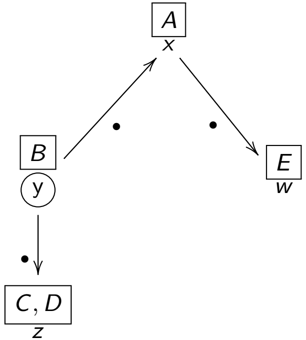
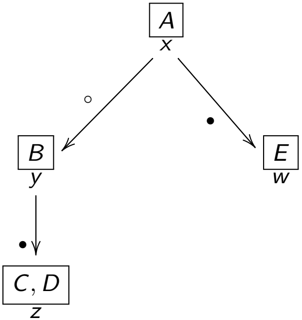
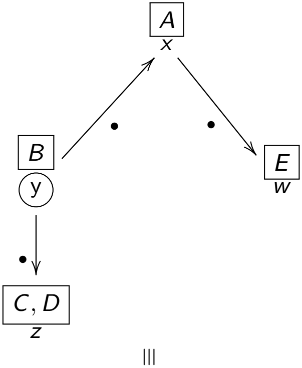
$Rxy, Rzy, Rwx, x : A, y : B, z : C, z : D, w : E$



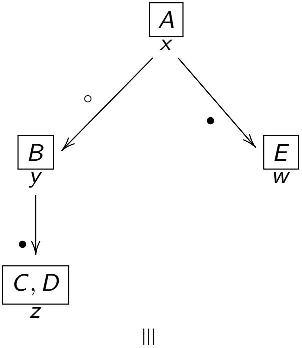
$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$



From Labelled to Display–A Problem?



$B, \bullet\{A, \bullet\{E\}\}, \bullet\{C, D\}$



$A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$

From Labelled to Display–A Problem?

- What is the relationship between

  $B, \bullet\{A, \bullet\{E\}\}, \bullet\{C, D\}$ and $A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}$?

From Labelled to Display–A Problem?

- What is the relationship between

$$B, \bullet\{A, \bullet\{E\}\}, \bullet\{C, D\} \text{ and } A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}?$$

- An observation:

$$\frac{B, \bullet\{A, \bullet\{E\}\}, \bullet\{C, D\}}{A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}} \text{(rp)} \qquad \frac{A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}}{B, \bullet\{A, \bullet\{E\}\}, \bullet\{C, D\}} \text{(rf)}$$

From Labelled to Display–A Problem?

- What is the relationship between

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- An observation:

$$\frac{B, \bullet\{A, \bullet\{E\}\}, \bullet\{C, D\}}{A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}} \text{(rp)} \qquad \frac{A, \circ\{B, \bullet\{C, D\}\}, \bullet\{E\}}{B, \bullet\{A, \bullet\{E\}\}, \bullet\{C, D\}} \text{(rf)}$$

Theorem

Regardless of the node chosen when translating from labelled polytree sequents to display sequents, the output will be display equivalent.

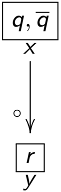
Translation Example: Display to Labelled

$$\dfrac{\dfrac{\dfrac{\dfrac{q, \overline{q}, \circ\{r\}}{q \lor \overline{q}, \circ\{r\}}}{\bullet\{q \lor \overline{q}\}, r}}{\blacksquare(q \lor \overline{q}), r}}{\blacksquare(q \lor \overline{q}) \lor r}$$

Translation Example: Display to Labelled

$$
\begin{array}{c}
\color{red}{q, \overline{q}, \circ\{r\}} \\
\hline
q \vee \overline{q}, \circ\{r\} \\
\hline
\bullet\{q \vee \overline{q}\}, r \\
\hline
\blacksquare(q \vee \overline{q}), r \\
\hline
\blacksquare(q \vee \overline{q}) \vee r
\end{array}
$$

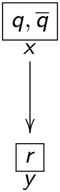
$q, \overline{q}, \circ\{r\}$



Translation Example: Display to Labelled

$$
\dfrac{\dfrac{\dfrac{\dfrac{q, \overline{q}, \circ\{r\}}{q \vee \overline{q}, \circ\{r\}}}{\bullet\{q \vee \overline{q}\}, r}}{\blacksquare(q \vee \overline{q}), r}}{\blacksquare(q \vee \overline{q}) \vee r}
$$

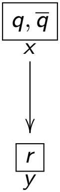
$q, \overline{q}, \circ\{r\}$



Translation Example: Display to Labelled

$$Rxy, x : q, x : \overline{q}, y : r$$

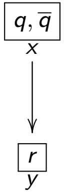
$$\cfrac{\cfrac{\cfrac{\cfrac{q, \overline{q}, \circ\{r\}}{q \lor \overline{q}, \circ\{r\}}}{\bullet\{q \lor \overline{q}\}, r}}{\blacksquare(q \lor \overline{q}), r}}{\blacksquare(q \lor \overline{q}) \lor r}$$



Translation Example: Display to Labelled

$$\dfrac{\dfrac{\dfrac{\dfrac{Rxy, x : q, x : \overline{q}, y : r}{q \vee \overline{q}, \circ\{r\}}}{\bullet\{q \vee \overline{q}\}, r}}{\blacksquare(q \vee \overline{q}), r}}{\blacksquare(q \vee \overline{q}) \vee r}$$

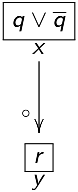
$Rxy, x : q, x : \overline{q}, y : r$



Translation Example: Display to Labelled

$$\cfrac{\cfrac{\cfrac{\cfrac{Rxy, x : q, x : \overline{q}, y : r}{q \lor \overline{q}, \circ\{r\}}}{\bullet\{q \lor \overline{q}\}, r}}{\blacksquare(q \lor \overline{q}), r}}{\blacksquare(q \lor \overline{q}) \lor r}$$

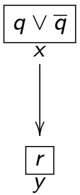
$q \lor \overline{q}, \circ\{r\}$



Translation Example: Display to Labelled

$$\cfrac{\cfrac{\cfrac{\cfrac{Rxy, x : q, x : \overline{q}, y : r}{q \vee \overline{q}, \circ\{r\}}}{\bullet\{q \vee \overline{q}\}, r}}{\blacksquare(q \vee \overline{q}), r}}{\blacksquare(q \vee \overline{q}) \vee r}$$

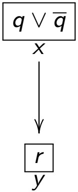
$q \vee \overline{q}, \circ\{r\}$



Translation Example: Display to Labelled

$$\dfrac{\dfrac{\dfrac{\dfrac{Rxy, x : q, x : \overline{q}, y : r}{q \vee \overline{q}, \circ\{r\}}}{\bullet\{q \vee \overline{q}\}, r}}{\blacksquare(q \vee \overline{q}), r}}{\blacksquare(q \vee \overline{q}) \vee r}$$

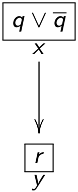
$$Rxy, x : q \vee \overline{q}, y : r$$



Translation Example: Display to Labelled

$$
\begin{array}{c}
Rxy, x : q, x : \overline{q}, y : r \\
\hline
\color{red}{Rxy, x : q \vee \overline{q}, y : r} \\
\hline
\bullet\{q \vee \overline{q}\}, r \\
\hline
\blacksquare(q \vee \overline{q}), r \\
\hline
\blacksquare(q \vee \overline{q}) \vee r
\end{array}
$$

$$Rxy, x : q \vee \overline{q}, y : r$$



Translation Example: Display to Labelled

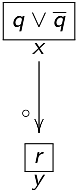
$$\begin{array}{c} Rxy, x : q, x : \overline{q}, y : r \\ \hline Rxy, x : q \vee \overline{q}, y : r \\ \hline \bullet\{q \vee \overline{q}\}, r \\ \hline \blacksquare(q \vee \overline{q}), r \\ \hline \blacksquare(q \vee \overline{q}) \vee r \end{array}$$

$\bullet\{q \vee \overline{q}\}, r$

Translation Example: Display to Labelled

$$\dfrac{\dfrac{\dfrac{\dfrac{Rxy, x : q, x : \overline{q}, y : r}{Rxy, x : q \vee \overline{q}, y : r}}{\bullet\{q \vee \overline{q}\}, r}}{\blacksquare(q \vee \overline{q}), r}}{\blacksquare(q \vee \overline{q}) \vee r}$$

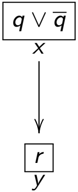
$\bullet\{q \vee \overline{q}\}, r$



Translation Example: Display to Labelled

$\bullet\{q \vee \overline{q}\}, r$

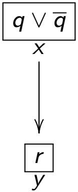
$$\dfrac{\dfrac{\dfrac{\dfrac{Rxy, x : q, x : \overline{q}, y : r}{Rxy, x : q \vee \overline{q}, y : r}}{\bullet\{q \vee \overline{q}\}, r}}{\blacksquare(q \vee \overline{q}), r}}{\blacksquare(q \vee \overline{q}) \vee r}$$



Translation Example: Display to Labelled

$$\dfrac{\dfrac{\dfrac{\dfrac{Rxy, x : q, x : \overline{q}, y : r}{Rxy, x : q \vee \overline{q}, y : r}}{\color{red}{\bullet\{q \vee \overline{q}\}, r}}}{\blacksquare(q \vee \overline{q}), r}}{\blacksquare(q \vee \overline{q}) \vee r}$$

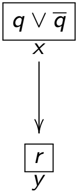
$Rxy, x : q \vee \overline{q}, y : r$



Translation Example: Display to Labelled

$$Rxy, x : q \lor \overline{q}, y : r$$

$$\begin{array}{c} Rxy, x : q, x : \overline{q}, y : r \\ \hline Rxy, x : q \lor \overline{q}, y : r \\ \hline \color{red}{Rxy, x : q \lor \overline{q}, y : r} \\ \hline \blacksquare(q \lor \overline{q}), r \\ \hline \blacksquare(q \lor \overline{q}) \lor r \end{array}$$



Translation Example: Display to Labelled

$$
\dfrac{\dfrac{\dfrac{\dfrac{Rxy, x : q, x : \overline{q}, y : r}{Rxy, x : q \vee \overline{q}, y : r}}{Rxy, x : q \vee \overline{q}, y : r}}{\blacksquare(q \vee \overline{q}), r}}{\blacksquare(q \vee \overline{q}) \vee r}
$$

$$\blacksquare(q \vee \overline{q}), r$$

$$\underset{x}{\boxed{\blacksquare(q \vee \overline{q}), r}}$$

Translation Example: Display to Labelled

$$\dfrac{\dfrac{\dfrac{\dfrac{Rxy, x : q, x : \overline{q}, y : r}{Rxy, x : q \vee \overline{q}, y : r}}{Rxy, x : q \vee \overline{q}, y : r}}{\color{red}{\blacksquare(q \vee \overline{q}), r}}}{\blacksquare(q \vee \overline{q}) \vee r}$$

$$x : \blacksquare(q \vee \overline{q}), x : r$$

$$\boxed{\blacksquare(q \vee \overline{q}), r}$$
$$x$$

Translation Example: Display to Labelled

$$\dfrac{\dfrac{\dfrac{\dfrac{Rxy, x : q, x : \overline{q}, y : r}{Rxy, x : q \vee \overline{q}, y : r}}{Rxy, x : q \vee \overline{q}, y : r}}{x : \blacksquare(q \vee \overline{q}), x : r}}{\blacksquare(q \vee \overline{q}) \vee r}$$

$$x : \blacksquare(q \vee \overline{q}), x : r$$

$$\underset{x}{\boxed{\blacksquare(q \vee \overline{q}), r}}$$

Translation Example: Display to Labelled

$$\blacksquare(q \lor \overline{q}) \lor r$$

$$\dfrac{\dfrac{\dfrac{\dfrac{Rxy, x : q, x : \overline{q}, y : r}{Rxy, x : q \lor \overline{q}, y : r}}{Rxy, x : q \lor \overline{q}, y : r}}{x : \blacksquare(q \lor \overline{q}), x : r}}{\blacksquare(q \lor \overline{q}) \lor r}$$

$$\boxed{\blacksquare(q \lor \overline{q}) \lor r}_{\,x}$$

Translation Example: Display to Labelled

$$Rxy, x : q, x : \overline{q}, y : r$$

$$Rxy, x : q \vee \overline{q}, y : r$$

$$Rxy, x : q \vee \overline{q}, y : r$$

$$x : \blacksquare(q \vee \overline{q}), x : r$$

$$\blacksquare(q \vee \overline{q}) \vee r$$

$$x : \blacksquare(q \vee \overline{q}) \vee r$$

$$\boxed{\blacksquare(q \vee \overline{q}) \vee r}_{\,x}$$

Translation Example: Display to Labelled

$$x : \blacksquare(q \vee \overline{q}) \vee r$$

$$\dfrac{\dfrac{\dfrac{\dfrac{Rxy, x : q, x : \overline{q}, y : r}{Rxy, x : q \vee \overline{q}, y : r}}{Rxy, x : q \vee \overline{q}, y : r}}{x : \blacksquare(q \vee \overline{q}), x : r}}{x : \blacksquare(q \vee \overline{q}) \vee r}$$

$$\underset{x}{\boxed{\blacksquare(q \vee \overline{q}) \vee r}}$$

Translation Example: Display to Labelled

$$\cfrac{\cfrac{\cfrac{\cfrac{Rxy, x : q, x : \overline{q}, y : r}{Rxy, x : q \lor \overline{q}, y : r}}{Rxy, x : q \lor \overline{q}, y : r}}{x : \blacksquare(q \lor \overline{q}), x : r}}{x : \blacksquare(q \lor \overline{q}) \lor r}$$

Translation Example: Display to Labelled

$$\dfrac{\dfrac{\dfrac{Rxy, x : q, x : \overline{q}, y : r}{Rxy, x : q \vee \overline{q}, y : r}}{x : \blacksquare(q \vee \overline{q}), x : r}}{x : \blacksquare(q \vee \overline{q}) \vee r}$$

Translation Example: Display to Labelled

$$
\begin{array}{c}
Rxy, x : q, x : \overline{q}, y : r \\ \hline
Rxy, x : q \vee \overline{q}, y : r \\ \hline
x : \blacksquare(q \vee \overline{q}), x : r \\ \hline
x : \blacksquare(q \vee \overline{q}) \vee r
\end{array}
\qquad \Longleftarrow \qquad
\begin{array}{c}
q, \overline{q}, \circ\{r\} \\ \hline
q \vee \overline{q}, \circ\{r\} \\ \hline
\bullet\{q \vee \overline{q}\}, r \\ \hline
\blacksquare(q \vee \overline{q}), r \\ \hline
\blacksquare(q \vee \overline{q}) \vee r
\end{array}
$$

Labelled Proofs to Display Proofs

- How to translate derivations in opposite direction?
  - More difficult
  - Not known for all general path axioms $\Pi p \rightarrow \Sigma p$
  - However, can be done with path axioms: $\Pi p \rightarrow \langle ? \rangle p$

Labelled Proofs to Display Proofs

- Goré, Postniece, and Tiu: *On the Correspondence between Display Postulates and Deep Inference in Nested Sequent Calculi for Tense Logics* (2011)
- Methodology:
  - 1. Introduce *propagation rules* for path axioms
  - 2. Show structural rules admissible
  - 3. Show every sequent in proof of new calculus is labelled polytree

Propagation rules

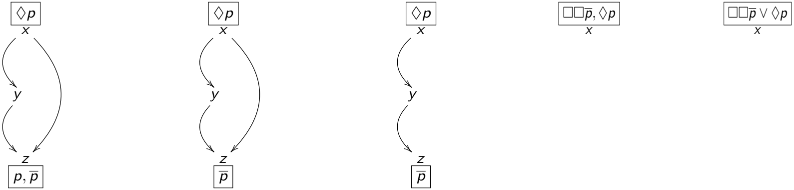
For the axiom $\sqcap p \to \langle ? \rangle p$:

$$\frac{\mathcal{R}, R_{\sqcap}xy, x : \langle ? \rangle A, y : A, \Gamma}{\mathcal{R}, R_{\sqcap}xy, x : \langle ? \rangle A, \Gamma} (Prop) \qquad \frac{\mathcal{R}, R_{\sqcap}xy, R_{\langle ? \rangle}xy, \Gamma}{\mathcal{R}, R_{\sqcap}xy, \Gamma} (Struc)$$

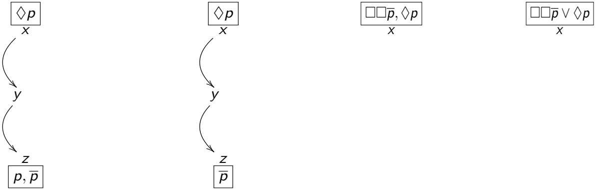
- Both give adequate labelled calculi for path extensions of Kt
- Both preserve nice proof-theoretic properties of base calculus
- Structural rule deletes relational atoms; propagation does not

Propagation vs. Structural

Deriving $\Box\Box\overline{p} \lor \Diamond p$ (Structural):



Deriving $\Box\Box\overline{p} \lor \Diamond p$ (Propagation):



The Resulting Calculus

Roadmap: Add propagation rules > Show structural rules admissible > Obtain cut-free complete calculus below:

$$\frac{}{\mathcal{R}, x : p, x : \overline{p}, \Gamma}\ (id)$$

$$\frac{\mathcal{R}, x : A, x : B, \Gamma}{\mathcal{R}, x : A \vee B, \Gamma}\ (\vee) \qquad \frac{\mathcal{R}, x : A, \Gamma \qquad \mathcal{R}, x : B, \Gamma}{\mathcal{R}, x : A \wedge B, \Gamma}\ (\wedge)$$

$$\frac{\mathcal{R}, Rxy, y : A, \Gamma}{\mathcal{R}, x : \Box A, \Gamma}\ (\Box)^{*} \qquad \frac{\mathcal{R}, Ryx, y : A, \Gamma}{\mathcal{R}, x : \blacksquare A, \Gamma}\ (\blacksquare)^{*}$$

$$\frac{\mathcal{R}, Rxy, y : A, x : \Diamond A, \Gamma}{\mathcal{R}, Rxy, x : \Diamond A, \Gamma}\ (\Diamond) \qquad \frac{\mathcal{R}, Ryx, y : A, x : \blacklozenge A, \Gamma}{\mathcal{R}, Ryx, x : \blacklozenge A, \Gamma}\ (\blacklozenge)$$

$$\frac{\mathcal{R}, R_{\Pi}xy, x : \langle ? \rangle A, y : A, \Gamma}{\mathcal{R}, R_{\Pi}xy, x : \langle ? \rangle A, \Gamma}\ (Prop)$$

The Resulting Calculus: Cycles

Example

$$\dfrac{\vdots}{Rxy, Ryx, \Gamma}$$

$$\vdots$$

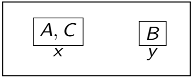
$$???$$

$$\dfrac{\vdots}{x : A}$$

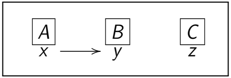
- Assume we have a derivation of a formula $x : A$.
- Assume a relational cycle exists in the derivation.
- Only rules that delete relational atoms are ($\blacksquare$) and ($\square$)
- Eigenvariable condition never satisfied in cycle.
- Therefore, cycle preserved downward.
- Therefore, $x : A$ contains a relational cycle (Contradiction)

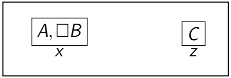
The Resulting Calculus: Disconnectivity

$$\frac{x : A, x : C, y : B}{x : A \vee C, y : B}\ (\vee)$$

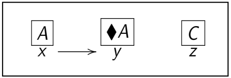


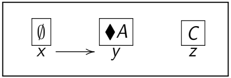
$$\frac{Rxy, x : A, y : B, z : C}{x : A, x : \Box B, y : B}\ (\Box)$$





$$\frac{Rxy, x : A, y : \blacklozenge A, z : C}{Rxy, y : \blacklozenge A, z : C}\ (\blacklozenge)$$





- Assume we have a derivation of a formula $x : A$.
- Assume a disconnected sequent exists in the derivation.
- All rules preserve disconnectivity downward.
- Therefore, $x : A$ is disconnected (Contradiction)

Labelled and Display "Equivalence"

Lemma

Every derivation of a formula in the labelled calculus with propagation rules contains solely labelled polytree sequents.

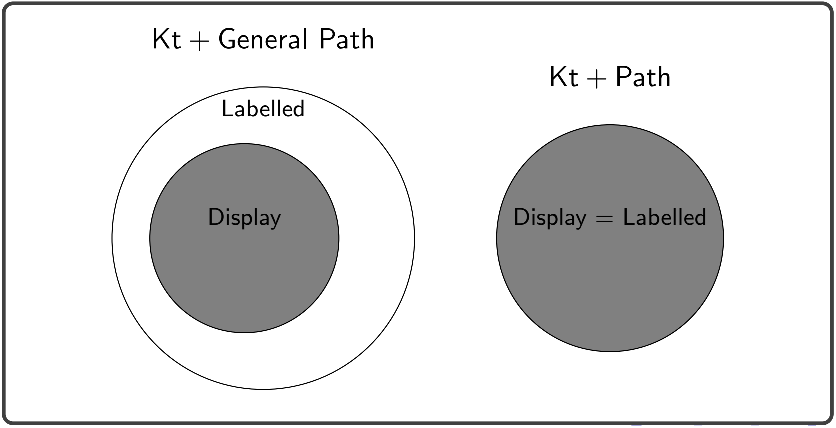
- Our translation translates labelled polytree sequents into display sequents and vice-versa.

Theorem

Display $=_{\mathsf{Kt}+Path}$ *Labelled*

Main Results

- General Path axioms: $\Pi p \to \Sigma p$ (with $\Pi, \Sigma \in \{\Diamond, \blacklozenge\}^*$)
- Path axioms: $\Pi p \to \langle ? \rangle p$ (with $\Pi \in \{\Diamond, \blacklozenge\}^*$ and $\langle ? \rangle \in \{\Diamond, \blacklozenge\}$)
- Path Axioms $\subset$ General Path Axioms



Application: Proof Search

- Goré, Postniece, and Tiu *On the Correspondence between Display Postulates and Deep Inference in Nested Sequent Calculi for Tense Logics* (2011)
- Tiu, Ianovski, Goré *Grammar Logics in Nested Sequent Calculus: Proof Theory and Decision Procedures* (2012)
- Deep-inference calculus has proof search
- Labelled calculus $\approx$ deep-inference calculus.
- *Internalizing* the labelled calculus generated a variant useful for proof search.
- Hope to get decidability by "internalizing" labelled calculi for other logics.

Future Work

- Can these translation methods be generalized to display and labelled calculi for other logics (such as bi-intuitionistic and intermediate logics)?
- How do we obtain translations from labelled to display for Kt extended with general path axioms (beyond path axioms)?
- What new proof-theoretic results can we extract from translations? What properties can be transferred between calculi?