Dynamic Approximation of Self-Referential Sentences as a Truthmaker

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We define a dynamic approximation of self-referential sentences, which for the Liar and the TruthTeller, generates three-valued Kleene logic, and allows us to obtain new 4- and 6-valued truth tables (Stepanov, 2021). We fix the self-referencing of the sentence using a special the self-referencing icon: $\mathbf{S}x$, which is placed in front of the predicate $P(x)$, which we call the core of the self-referential sentence. A self-referential sentence looks like this: $\mathbf{S}xP(x)$. (1) Expression (1) obeys the axiom of self-reference (Feferman 1984): $\mathbf{S}xP(x) \leftrightarrow P(\mathbf{S}xP(x))$. (2) Peirce (Emily, 1975) applied (2) to infinite Liar sentence: $\mathbf{S}xP(x) \leftrightarrow P(P(P(...\mathbf{S}xP(x)...)))$. (3) Let's break it down into iterative steps: $\mathbf{S}xP(x) \approx \mathsf{S}xP(x) =< x, P(x), P(P(x)), \dots >$. (4) Expression (4) on the right will be considered as an approximation $\approx$ of a real self-referential sentence $\mathbf{S}xP(x)$. To denote the result of the approximation the sign $\mathsf{S}xP(x)$. Expression (4) is the definition of the trajectory of a dynamical system of the form $(\{0,1\}, P(x))$ with orbits $< P^n(x), n \in Z^+ >$, where $P^n(x) = P(P^{n-1}(x))$, (Konev and etc., 2006). Consider the case when the kernels of selfref sentences $P(x)$ are composed of $Tr(x)$ using propositional connectives equivalence and negation: $P(x) \in \{Tr(x), \neg Tr(x), Tr(x) \leftrightarrow Tr(x), Tr(x) \leftrightarrow \neg Tr(x)\}$. (5) It is easy to see that expression (4) is periodic, with a maximum period of 2. This means that the second and third terms of the sequence (4) determine the entire remaining infinite sequence. Therefore, in our case, we rightfully shorten the definition of a self-referencing quantifier as follows: $\mathsf{S}xP(x) = < x, P(x), P(P(x)) >$. (6) The variable x and the predicates $P(x)$ from (5) in our case take values from $\{0,1\}$. **Definition1**: For $\mathsf{S}xP(x) \rightleftharpoons \{< 1, P(1), P(P(1)) > , < 0, P(0), P(P(0) >\}$. (7)

This is the table for the negation symbol:

$\mathsf{S}xP(x)$	$\neg\mathsf{S}xP(x)$
$\{< 1,1,1 >; < 0,1,1 >\} = T$	$F = \{< 1,0,0 >; < 0,0,0 >\}$ (False)
$\{< 1,0,1 >; < 0,1,0 >\} = A$	$A = \{< 0,1,0 >; < 1,0,1 >\}$ (Antinomy, Liar)
$\{< 1,1,1 >; < 0,0,0 >\} = V$	$V = \{< 0,0,0 >; < 1,1,1 >\}$ (Void,TruthTeller)
$\{< 1,0,0 >; < 0,0,0 >\} = F$	$T = \{< 1,0,0 >; < 0,0,0 >\}$ (True)

We define two-place connectives $o \in \{\wedge, \vee, \rightarrow, \leftarrow, \leftrightarrow\}$ for two S-formulas $\mathsf{S}xP(x)$ and $\mathsf{S}xQ(x)$. **Definition2**: We study such a variant of two-place connectives, when the trajectories of estimates of the formula $\mathsf{S}xP(x)$ of the one branch (x = 1 or x = 0) interact with the trajectories of the formula $\mathsf{S}xQ(x)$ of the same branch ($x = 1$ or $x = 0$, respectively):

$$\mathsf{S}xP(x) \text{ o } \mathsf{S}xQ(x) \rightleftharpoons$$
$$\{< 1, P(1), P(P(1)) >, < 0, P(0), P(P(0) >\}o\{< 1, Q(1), Q(Q(1)) >, < 0, Q(0), Q(Q(0)) >\} =$$
$$\{< 1, P(1), P(P(1)) > o < 1, Q(1), Q(Q(1)) >, < 0, P(0), P(P(0)) > o < 0, Q(0), Q(Q(0)) >\} =$$
$$\{< 1o1, P(1)oQ(1), P(P(1))oQ(Q(1)) >, < 0o0, P(0)oQ(0), P(P(0))oQ(Q(0)) >\}.$$

Exm.: $F \wedge V = \{<1,0,0>, <0,0,0>\} \wedge \{<1,1,1>, <0,0,0>\} = \{<1,0,0>, <0,0,0>\}$=F.
Let's compare Kleene-Priest tables with ours tables on our rules for A and V:

Kleene-Priest p

∧	t	p	f
t	t	p	f
p	p	p	f
f	f	f	f

≡

Hypothesis: p = A

∧	T	A	F
T	T	A	F
A	A	A	F
F	F	F	F

≡

Hypothesis: p = V

∧	T	V	F
T	T	V	F
V	V	V	F
F	F	F	F

Lemma 1: 1. The sentences *Liar* (A) have the tabular model, coinciding with tabular model *Liar* (p) of Priest (Priest, 1979) and, accordingly, the same evidential theory.
2. The sentences *TruthTeller* (V) have the same configuration tabular model, coinciding with configuration tabular model *Liar* (p) of Priest (Priest, 1979).

Our table

∧	T	A	V	F
T	T	A	V	F
A	A	A	av	F
V	V	av	V	F
F	F	F	F	F

Our table

∨	T	A	V	F
T	T	T	T	T
A	T	A	va	A
V	T	va	V	V
F	T	A	V	F

‖

(Dunn, 2019)

∨	T	2	3	F
T	T	T	T	T
2	T	2	T	2
3	T	T	3	3
F	T	2	3	F

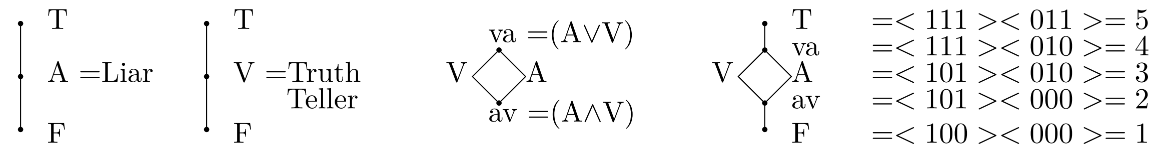
Lemma 2: When constructing tables for the interaction of V and A, new truth values were obtained: A∧V=$\{<1,0,1>, <0,0,0>\}$=av=¬(va).A∨V=$\{<1,1,1>, <0,1,0>\}$=va=¬(av).

¬	
T	F
va	av
A	A
V	V
av	va
F	T

∧	T	va	A	V	av	F
T	T	va	A	V	av	F
va	va	va	A	V	av	F
A	A	A	A	av	av	F
V	V	V	av	V	av	F
av	av	av	av	av	av	F
F	F	F	F	F	F	F

∨	T	va	A	V	av	F
T	T	T	T	T	T	T
va	T	va	va	va	va	va
A	T	va	A	va	A	A
V	T	va	va	V	V	V
av	T	va	A	V	av	av
F	T	va	A	V	av	F

Lemma 3: The next four lattices are DeMorgan lattices, a la (Leitgeb, 1999):
$\{ F \leq av \leq A \leq V \leq va \leq T \} : \{ 1 \leq 2 \leq 3 \leq 3 \leq 4 \leq 5 \} :$



DM3A **DM3V** **DM4VA** **DM6**

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